Coset approach to supersymmetric component actions

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It is a well known fact that a domain wall spontaneously breaks the Poincaré invariance of the target space down to the symmetry group of the world volume subspace.
This breaking results in the appearing of the Goldstone bosons associated with spontaneously broken symmetries.
When we are dealing with the purely bosonic p-branes this information is enough to construct the corresponding action.
The situation with the bosonic D-branes, which necessarily contain the gauge fields is less clear, despite the knowledge of the explicit actions, etc.

Fortunately, in the supersymmetric cases, where the supersymmetry is also partially spontaneously broken, the bosonic sector, which is the combination of Nambu-Goto and Born-Infeld actions, appears automatically.

From the mathematical point of view, the most appropriate approach to describe a partial breaking of Poincaré symmetry is the method of the nonlinear realization. Schematically, this approach works as follows.
After splitting the generators of the target space D-dimensional Poincaré group, which is supposed to be spontaneously broken on the world volume down to the d-dimensional Poincaré subgroup, into the generators of unbroken $\{P, M\}$ and spontaneously broken $\{Z, K\}$ symmetries, where the generators P and Z form D-dimensional translations, M generators span the $so(1, d-1)$ - Lorentz algebra on the world volume, while generators K belong to the coset $so(1, D-1)/so(1, d-1) \times so(D-d)$, one may realize all the transformations of D-dimensional Poincaré group by the left action on the coset element

$$g = e^{xP} e^{q(x)Z} e^{\Lambda(x)K}.$$

The spontaneous breaking of Z and K symmetries is reflected in the character of corresponding coset coordinates which are Goldstone fields $q(x)$ and $\Lambda(x)$ in the present case. The transformation properties of coordinates x and fields $q(x), \Lambda(x)$ may be easily found in this approach, while all information about geometric properties is contained in the Cartan forms

$$g^{-1}dg = \Omega_P P + \Omega_M M + \Omega_Z Z + \Omega_K K \,.$$

All Cartan forms except for Ω_M are transformed homogeneously under all symmetries.

Due to the general theorem (Inverse Higgs Phenomenon) not all of the above Goldstone fields have to be treated as independent. In the present case the fields $\Lambda(x)$ can be covariantly expressed through x-derivatives of $q(x)$ by imposing the constraint

$$\Omega_Z = 0.$$

Thus, we are dealing with the fields $q(x)$ only. It is very important that the form Ω_P defines the vielbein $\mathcal{E}$ (d-bein in the present case), connecting the covariant world volume coordinate differentials Ω_P and the world volume coordinate differential dx as

$$\Omega_P = \mathcal{E} \cdot dx.$$

Combining all these ingredients, one may immediately write the action

$$S = -\int d^d x + \int d^d x \ \det \mathcal{E},$$

which is invariant under all symmetries. In the action we have added the trivial first term to fulfill the condition $S_{q=0} = 0$. This action is just the static gauge form of the actions of $p = (d-1)$-branes.

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The $D = 4$ Poincaré group can be realized in its coset over the $d = 3$ Lorentz group $SO(1,2)$ as

$$g = e^{x^{ab}P_{ab}} e^{q(x)Z} e^{\Lambda^{ab}(x)K_{ab}} .$$

To reduce the number of independent superfields one has to impose the constraints

$$\Omega_Z = 0 \quad \Rightarrow \quad \partial_{ab}q + \frac{4}{1+2\lambda^2}\lambda_{ab} = 0, \quad \lambda_{ab} = -\frac{1}{2}\,\frac{\partial_{ab}q}{1+\sqrt{1-\partial q\cdot\partial q}}.$$

So, the bosonic field $q(x)$ is the only essential Goldstone field we need for this case of the partial breaking of the $D = 4$ Poincaré symmetry.
The expression for vielbein $\mathcal{E}$ follows from the Cartan form Ω_P

$$\Omega_P = \left(dx^i - \frac{4}{1+2\lambda^2}\lambda^i\lambda_j dx^j\right) \quad \Rightarrow \quad \mathcal{E}_i^j = \delta_i^j - \frac{4\lambda_i\lambda^j}{1+2\lambda^2}.$$

Thus,

$$\det\mathcal{E} = \frac{1-2\lambda^2}{1+2\lambda^2} = \sqrt{1-\frac{1}{2}\partial q\cdot\partial q} \to S = \int d^3x\left(1-\sqrt{1-\frac{1}{2}\partial q\cdot\partial q}\right).$$

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The supersymmetric generalization of the coset approach involves into the game new spinor generators Q and S which extend the D-dimensional Poincaré group to the supersymmetric one

$$\{Q,Q\} \sim P, \; \{S,S\} \sim P, \; \{Q,S\} \sim Z\,.$$

The most interesting cases are those when the Q supersymmetry is kept unbroken, while the S supersymmetry is supposed to be spontaneously broken. When $\#Q = \#S$ we are facing the so called $1/2$ Partial Breaking of Global Supersymmetry cases, which most of all interesting supersymmetric domain walls belong to. Only such cases of supersymmetry breaking will be considered in this talk.
Now, all symmetries can be realized by group elements acting on the coset element

$$g = e^{xP} e^{\theta Q} e^{q(x,\theta)Z} e^{\psi(x,\theta)S} e^{\Lambda(x,\theta)K}\,.$$

The main novel feature of the supersymmetric coset is the appearance of the Goldstone superfields $\{q(x,\theta), \psi(x,\theta), \Lambda(x,\theta)\}$.

The rest of the coset approach machinery works in the same manner: one may construct the Cartan forms (which will contain the new forms Ω_Q and Ω_S),
one may find the supersymmetric d-bein and corresponding bosonic ∇_P and spinor ∇_Q covariant derivatives, etc.
One may even write the proper generalizations of the covariant constraints.
Unfortunately, this similarity between purely bosonic and supersymmetric cases is not complete due to the existence of the following important new features of theories with partial breaking of global supersymmetry:

- In contrast with the bosonic case, not all of the physical fields appear among the parameters of the coset. Nevertheless, *it is true* that the *all physical bosonic components* can be found in the quantity $\nabla_Q \psi|$.
- The supersymmetric generalization of the bosonic kinematic constraints in most cases contains not only kinematic conditions, but also dynamic superfield equations of motion and in many cases it is unknown how to split these constraints into kinematical and dynamical ones.

Therefore, all that we can do until now, within the supersymmetric coset approach, is

- to find the transformation properties of the superfields and construct the covariant derivatives
- to find the superfield equations of motion and/or covariant variants of irreducibility constraints.

That is why during recent years some new methods to construct the actions (in terms of superfield or in terms of physical components) have been proposed. Among them one should mention the construction of the linear realization of partially broken supersymmetry and reduction from higher dimensional supersymmetric D-brane action to lower dimensions.

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The main idea is to start with the Ansatz for the action manifestly invariant with respect to spontaneously broken supersymmetry. Funny enough, it is rather easy to do, due to the following properties:

- in our parametrization of the coset element the superspace coordinates do not transform under broken supersymmetry. Thus, all components of superfields transform independently,
- the covariant derivatives ∇_P and ∇_Q are invariant under broken supersymmetry. Therefore, the bosonic physical components which are contained in $\nabla_Q \psi|$ can be treated as "matter fields" with respect to broken supersymmetry,
- all physical fermionic components are just $\theta = 0$ projections of the superfield $\psi(x, \theta)$ and these components transform as the fermions of the Volkov-Akulov model with respect to broken supersymmetry.

The immediate consequence of these facts is the conclusion that the physical fermionic components can enter the component on-shell action through the determinant of the d-bein $\mathcal{E}$ constructed with the help of the Cartan form Ω_P in the limit $\theta = 0$, or through the space-time ∇_P derivatives of the "matter fields".
Thus, the most general Ansatz for the on-shell component action, which is invariant with respect to spontaneously broken supersymmetry, has the form

$$S = \int d^d x - \int d^d x \det \mathcal{E} \mathcal{F}(\nabla_Q \psi|, \nabla_P q|).$$

Note, that the arguments of the function $\mathcal{F}$ are the bosonic physical components $\nabla_Q \psi|$ and covariant space-time derivatives of q (which, by the way, are also contained in $\nabla_Q \psi|$).

The explicit form of the function $\mathcal{F}$ can be fixed by two additional requirements

- The action should have a proper bosonic limit, which is known in almost all interesting cases. One should note, that this limit for our action is trivial
$$S_{bos} = \int d^d x \left(1 - \mathcal{F}(\nabla_Q \psi|, \partial_P q)\right).$$
- The action in the linear limit should possess a linear version of unbroken supersymmetry, i.e. it should be just a sum of the kinetic terms for all bosonic and fermionic components with the relative coefficients fixed by unbroken supersymmetry.

These conditions completely fix the component action. Of course, as the final step, the invariance with respect to unbroken supersymmetry has to be checked.

Outline

The $N = 1, D = 4$ super Poincaré group can be realized in its coset over the $d = 3$ Lorentz group $SO(1,2)$

$$g = e^{x^{ab}P_{ab}} e^{\theta^a Q_a} e^{qZ} e^{\psi^a S_a} e^{\Lambda^{ab} K_{ab}} \; .$$

Here, (x^{ab}, θ^a) are $N = 1, d = 3$ superspace coordinates, while the remaining coset parameters are Goldstone superfields:

$$\psi^a \equiv \psi^a(x, \theta), \;\; q \equiv q(x, \theta), \;\; \Lambda^{ab} \equiv \Lambda^{ab}(x, \theta).$$

To reduce the number of independent superfields one has to impose the constraints:

$$\Omega_Z = 0 \quad \Rightarrow \quad \left\{ \begin{array}{ll} \nabla_{ab} q + \frac{4}{1+2\lambda^2} \lambda_{ab} = 0 & \text{(a)} \\ \nabla_a q - \psi_a = 0 & \text{(b)} \end{array} \right.$$

These equations allow to express $\lambda_{ab}(x, \theta)$ and $\psi^a(x, \theta)$ through covariant derivatives of $q(x, \theta)$. Thus, the bosonic superfield $q(x, \theta)$ is the only essential Goldstone superfield we need for this case of the partial breaking of the global supersymmetry. These constraints are covariant under all symmetries and they do not imply any dynamics and leave $q(x, \theta)$ off-shell.

The last step we can make within the coset approach is to write the covariant superfield equations of motion as:

$$\Omega_S| = 0 \quad \Rightarrow \quad \text{(a)} \;\; \nabla^a \psi_a = 0 \,, \quad \text{(b)} \;\; \nabla_{(a}\psi_{b)} = -2\lambda_{ab} \,.$$

where $|$ denotes the ordinary $d\theta$- projection of the form Ω_S.
These equations imply the proper dynamical equation of motion

$$\nabla^a \nabla_a q = 0,$$

with the bosonic limit (for $q(x) \equiv q(x,\theta)|_{\theta=0}$)

$$\partial_{ab} \left(\frac{\partial^{ab} q}{\sqrt{1 - \frac{1}{2}\partial q \cdot \partial q}} \right) = 0 \,,$$

which corresponds to the "static gauge" form of the $D = 4$ membrane Nambu-Goto action

$$S = \int d^3x \left(1 - \sqrt{1 - \frac{1}{2}\partial^{ab} q \partial_{ab} q} \right) \,.$$

Outline

Funny enough, if we will be interested in the component action, then it can be constructed almost immediately within the nonlinear realization approach. One may check that all important features of the on-shell (i.e. with the equations taken into account) component action we summarize in the Introduction, are present in the case at hands. Indeed,

- all physical components, i.e. $q|_{\theta=0}$ and $\psi^a|_{\theta=0}$, are among the "coordinates" of our coset as the $\theta = 0$ parts of the corresponding superfields,
- under spontaneously broken supersymmetry the superspace coordinates θ^a do not transform at all. Therefore, the corresponding transformation properties of the fermionic components $\psi^a|_{\theta=0}$ are the same as in the Volkov-Akulov model , where all supersymmetries are supposed to be spontaneously broken,
- Finally, the $\theta = 0$ component of our essential Goldstone superfield $q(x, \theta)$ does not transform under spontaneously broken supersymmetry and, therefore, it behaves like a "matter" field within the Volkov-Akulov scheme.

As the immediate consequences of these features we conclude that

- The fermionic components $\psi^a|_{\theta=0}$ may enter the component action either through $\det \mathcal{E}$ or through the covariant derivatives ∇_{ab}, only,
- The "matter" field – $q|_{\theta=0}$ may enter the action only through covariant derivatives $\nabla_{ab} q$.

Thus, the unique candidate to be the component on-shell action, invariant with respect to spontaneously broken supersymmetry S reads

$$S = \alpha \int d^3x + \beta \int d^3x \det \mathcal{E} \mathcal{F}(\nabla^{ab} q \nabla_{ab} q),$$

with an arbitrary, for the time being, function $\mathcal{F}$.
All other interactions between the bosonic component q and the fermions of spontaneously broken supersymmetry ψ^a are forbidden!

This action is the most general component action invariant with respect to unbroken supersymmetry. But in the present case we explicitly know its bosonic limit - it should be just the Nambu-Goto action. Moreover, the linearized form of the action, in according with its invariance with respect to unbroken supersymmetry, has to be

$$S_{lin} \sim \psi^a \partial_{ab} \psi^b - \frac{1}{4} \partial^{ab} q \partial_{ab} q .$$

Combining all these ingredients, which completely fix the parameters α and β, we can write the component action of $N = 1, D = 4$ supermembrane as

$$S = \int d^3 x \left[2 - \det \mathcal{E} \left(1 + \sqrt{1 - \frac{1}{2} \nabla^{ab} q \nabla_{ab} q} \right) \right] .$$

The explicit expression for $\det \mathcal{E}$ has the form

$$\det \mathcal{E} = 1 + \frac{1}{2} \psi^a \partial_{ab} \psi^b + \frac{1}{8} \psi^d \psi_d \left(\partial^{ab} \psi_b \partial_{ac} \psi^c + \frac{1}{2} \partial^{ab} \psi^c \partial_{ab} \psi_c \right) ,$$

and

$$\nabla_{ab} = (\mathcal{E}^{-1})^{cd}_{ab} \, \partial_{cd} \; , \; \mathcal{E}^{cd}_{ab} = \frac{1}{2} (\delta^c_a \delta^d_b + \delta^d_a \delta^c_b) + \frac{1}{4} (\psi^c \partial_{ab} \psi^d + \psi^d \partial_{ab} \psi^c).$$

Let us stress, that such a simple form of the component action is achieved only in the rather specific basis, where the bosonic q and fermionic fields ψ^a are the Goldstone fields of the nonlinear realization.
Surely, this choice is not unique and in different bases the explicit form of action could drastically change.
The detailed proof that the constructed action is invariant with respect to both, broken and unbroken supersymmetries, can be found in our paper.

Outline

Due to the duality between scalar field and gauge field strength in $d = 3$, the action for $D2$-brane can be easily constructed within the coset approach. The idea of the construction is similar to the purely bosonic case. The crucial step is to treat the first, bosonic component of λ_{ab} as an independent component (i.e. to ignore the constraint $\Omega_Z = 0$). Thus, the generalized variant of the action reads

$$S = \int d^3x \left[2 - \det \mathcal{E} - \det \mathcal{E} \left(1 + 2 \frac{\lambda^{ab}(\nabla_{ab} q + 2\lambda_{ab})}{1 - 2\lambda^2} \right) \right].$$

All these summands have a description in terms of $\theta = 0$ parts of the Cartan forms. The first term is just a volume form constructed from ordinary differentials dx^{ab}. The second terms is a volume form constructed from semi-covariant differentials $d\hat{x}^{ab}$

$$d\hat{x}^{ab} = dx^{ab} + \frac{1}{2}\psi^a \partial_{cd} \psi^b dx^{cd}.$$

Finally, the last term is a volume form constructed from $\theta = 0$ component of the forms Ω_P^{ab}

$$d\tilde{x}^{ab} = d\hat{x}^{ab} + \frac{2}{1 - 2\lambda^2} \lambda^{ab} \left(\nabla_{cd} q + 2\lambda_{cd} \right) d\hat{x}^{cd}.$$

Since the action depends only on λ^{ab} and not on its derivatives, the λ-equation of motion

$$\nabla_{ab} q = -\frac{4\lambda_{ab}}{1+2\lambda^2}$$

can be used to eliminate λ^{ab} in favor of $\nabla_{ab} q$. Clearly, this equation is just the (a) part of the constraints $\Omega_Z = 0$, we ignored while introducing the action. Plugging λ expressed through q back into action gives us the action for supermembrane.
Alternatively, the equation of motion for q

$$\partial_{ab} \left[\frac{\det \mathcal{E} \, \lambda^{cd} \, \left(\mathcal{E}^{-1}\right)_{cd}{}^{ab}}{1-2\lambda^2} \right] = 0$$

has the form of the $d = 3$ Bianchi identity for the field strength F^{ab}

$$F^{ab} \equiv \frac{\det \mathcal{E} \, \lambda^{cd} \, \left(\mathcal{E}^{-1}\right)_{cd}{}^{ab}}{1-2\lambda^2} \qquad \Rightarrow \qquad \partial_{ab} F^{ab} = 0.$$

Substituting this into the action and integrating by parts, one may bring it to the supersymmetric *D*2-brane action

$$S = \int d^3x \left[2 - \det \mathcal{E} \left(1 + \sqrt{1 + 8\widetilde{F}^2} \right) \right]$$

where

$$\widetilde{F}_{ab} \equiv \frac{(\mathcal{E})_{ab}{}^{cd} F_{cd}}{\det \mathcal{E}} = \frac{\lambda_{ab}}{1 - 2\lambda^2}.$$

Therefore,

$$S = 2 \int d^3x \left[1 - \det \mathcal{E} \, \frac{1}{1 - 2\lambda^2} \right].$$

Clearly, in the bosonic limit $\widetilde{F}_{ab} = F_{ab}$ and thus, the bosonic part of this action is the standard Born-Infeld action for *D*2-brane, as it should be.

Outline

It is well known fact that the action for the given pattern of the supersymmetry breaking is completely defined by the choice of the corresponding Goldstone supermultiplet. The bosonic scalars in the supermultiplet are associated with the central charges in the supersymmetry algebra. Thus, for the system with chiral supermultiplet one has to choose $N = 16, d = 1$ Poincaré superalgebra with two central charges

$$\left\{Q^{ia}, \overline{Q}_{jb}\right\} = 2\delta^a_b\delta^i_j P\,, \quad \left\{S^{ia}, \overline{S}_{jb}\right\} = 2\delta^a_b\delta^i_j P\,,$$

$$\left\{Q^{ia}, S^{jb}\right\} = 2i\epsilon^{ab}\epsilon^{ij} Z\,, \quad \left\{\overline{Q}_{ia}, \overline{S}_{jb}\right\} = -2i\epsilon_{ab}\epsilon_{ij}\overline{Z}\,.$$

Here, $Q^{ia}, \overline{Q}_{ia}$ and $S^{ia}, \overline{S}_{ia}$ are the generators of unbroken and spontaneously broken $N = 8$ supersymmetries, respectively. P is the generator of translations, while Z and $\overline{Z}$ are the central charges generators . The indices $i, a = 1, 2$ refer to the indices of the fundamental representations of the two commuting $SU(2)$ groups.

In the coset approach the statement that S supersymmetry and $Z, \overline{Z}$ translations are spontaneously broken is reflected in the structure of the group element g

$$g = e^{itP}\, e^{\theta_{ia}Q^{ia}+\bar\theta^{ia}\overline{Q}_{ia}}\, e^{i(qZ+\bar q\overline{Z})}\, e^{\psi_{ia}S^{ia}+\bar\psi^{ia}\overline{S}_{ia}}.$$

Once we state that the coordinates ψ and q are the superfields depending on the superspace coordinates $\{t,\theta,\bar\theta\}$, then we are dealing with the spontaneously breaking of the corresponding symmetries. Thus, in the present case we treat $\psi(t,\theta,\bar\theta), q(t,\theta,\bar\theta)$ as $N = 8$ Goldstone superfields accompanying $N = 16 \to N = 8$ breaking of supersymmetry.

The transformation properties of the coordinates and superfields under the unbroken and broken supersymmetries have the following form

$$\delta_Q t = i\left(\epsilon_{ia}\bar{\theta}^{ia} + \bar{\epsilon}^{ia}\theta_{ia}\right), \quad \delta_Q \theta_{ia} = \epsilon_{ia}, \quad \delta_Q \bar{\theta}^{ia} = \bar{\epsilon}^{ia},$$

$$\delta_S t = i\left(\eta_{ia}\bar{\psi}^{ia} + \bar{\eta}^{ia}\psi_{ia}\right), \quad \delta_S \psi_{ia} = \eta_{ia}, \quad \delta_S \bar{\psi}^{ia} = \bar{\eta}^{ia},$$
$$\delta_S q = -2\eta_{ia}\theta^{ia}, \quad \delta_S \bar{q} = 2\bar{\eta}^{ia}\bar{\theta}_{ia}.$$

The local geometric properties of the system are specified by the left-invariant Cartan forms

$$g^{-1}dg = i\omega_P P + (\omega_Q)_{ia}Q^{ia} + (\bar{\omega}_Q)^{ia}\overline{Q}_{ia} + i\omega_Z Z + i\bar{\omega}_Z\overline{Z} + (\omega_S)_{ia}S^{ia} + (\bar{\omega}_S)^{ia}\overline{S}_{ia}$$

which look extremely simple in our case:

$$\omega_P = dt - i(\bar{\theta}^{ia}d\theta_{ia} + \theta_{ia}d\bar{\theta}^{ia} + \bar{\psi}^{ia}d\psi_{ia} + \psi_{ia}d\bar{\psi}^{ia}),$$

$$(\omega_Q)_{ia} = d\theta_{ia}, \quad (\bar{\omega}_Q)^{ia} = d\bar{\theta}^{ia}, \quad (\omega_S)_{ia} = d\psi_{ia}, \quad (\bar{\omega}_S)^{ia} = d\bar{\psi}^{ia},$$
$$\omega_Z = dq + 2\psi^{ia}d\theta_{ia}, \quad \bar{\omega}_Z = d\bar{q} - 2\bar{\psi}_{ia}d\bar{\theta}^{ia}.$$

Using the covariant differentials $\{\omega_P, d\theta_{ia}, d\bar{\theta}^{ia}\}$, one may construct the covariant derivatives

$$\partial_t = E\nabla_t\,, \quad E = 1 - i\left(\psi_{ia}\dot{\bar{\psi}}^{ia} + \bar{\psi}^{ia}\dot{\psi}_{ia}\right), \quad E^{-1} = 1 + i\left(\psi_{ia}\nabla_t\bar{\psi}^{ia} + \bar{\psi}^{ia}\nabla_t\psi_{ia}\right),$$

$$\nabla^{ia} = D^{ia} - i\left(\psi_{kb}D^{ia}\bar{\psi}^{kb} + \bar{\psi}^{kb}D^{ia}\psi_{kb}\right)\nabla_t\,,$$

$$\overline{\nabla}_{ia} = \overline{D}_{ia} - i\left(\psi_{kb}\overline{D}_{ia}\bar{\psi}^{kb} + \bar{\psi}^{kb}\overline{D}_{ia}\psi_{kb}\right)\nabla_t\,.$$

Finally, one may reduce the number of independent Goldstone superfields by imposing the conditions on the $d\theta$- projections of the Cartan forms $\omega_Z, \bar{\omega}_Z$

$$\omega_Z|_\theta = 0 \quad \Rightarrow \quad \overline{\nabla}_{ia}q = 0, \quad \nabla^{ia}q - 2\psi^{ia} = 0,$$

$$\overline{\omega}_Z|_\theta = 0 \quad \Rightarrow \quad \nabla^{ia}\bar{q} = 0, \quad \overline{\nabla}_{ia}\bar{q} + 2\bar{\psi}_{ia} = 0.$$

These constraints are purely kinematical ones. They impose the covariant chirality conditions on the superfields q and $\bar{q}$ and in addition they express the fermionic Goldstone superfields $\psi^{ia}, \bar{\psi}_{ia}$ as the spinor derivatives of the q and $\bar{q}$, thereby realizing the inverse Higgs phenomenon.

Outline

The chirality conditions are not enough to select irreducible $N = 8$ supermultiplet: one has impose additional, second order in the spinor derivatives constraints on the superfield $\{q, \bar{q}\}$. Unfortunately, as it is often happened in the coset approach, the direct covariantization of the irreducibility constraints is not covariant, while the simultaneous covariantization of the constraints and the equations of motion works perfectly. That is why we propose the following equations which should describe our superparticle

$$\nabla^{ia}\psi_{jb} = 0, \quad \overline{\nabla}_{ia}\bar{\psi}^{jb} = 0.$$

These equations are covariant with respect to both unbroken and broken supersymmetries.

At this point one should wonder whether these equations are self-consistent?

Indeed, due to the eqs. $\Omega_Z = 0, \;\; \overline{\Omega}_Z = 0$ one finds

$$\nabla^{ia}\psi_{jb} = \frac{1}{2}\nabla^{ia}\nabla_{jb}q = 0 \quad \Rightarrow \quad \{\nabla^{ia}, \nabla_{jb}\}q = 0.$$

So, one may expect some additional conditions on the superfield q due to the relations of covariant derivatives. However, when the constraints $\nabla^{ia}\psi_{jb} = 0$, $\overline{\nabla}_{ia}\bar{\psi}^{jb} = 0$ are taken into account, we have

$$\left\{\nabla^{ia}, \nabla^{jb}\right\} = 0, \qquad \left\{\overline{\nabla}_{ia}, \overline{\nabla}_{jb}\right\} = 0.$$

Thus the equations are self-consistent.
One should note that the rest of the commutators are also simplified as

$$\left\{\nabla^{ia}, \overline{\nabla}_{jb}\right\} = -2i\delta^i_j\delta^a_b(1+\lambda\bar{\lambda})\nabla_t, \quad \left[\nabla_t, \nabla^{ia}\right] = 2i\bar{\lambda}\nabla_t\psi^{ia}\nabla_t,$$

where we introduced the superfields $\{\lambda, \bar{\lambda}\}$

$$\left\{ \begin{array}{l} \overline{\nabla}_{ia}\psi_{jb} + \epsilon_{ij}\epsilon_{ab}\lambda = 0 \\ \nabla^{ia}\bar{\psi}^{jb} + \epsilon^{ij}\epsilon^{ab}\bar{\lambda} = 0 \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \nabla_t q + \frac{i\lambda}{1+\lambda\bar{\lambda}} = 0 \\ \nabla_t \bar{q} - \frac{i\bar{\lambda}}{1+\lambda\bar{\lambda}} = 0 \end{array} \right.$$

Finally, one may easily find that the spinor equations

$$\nabla^{ia}\psi_{jb} = 0, \quad \overline{\nabla}_{ia}\bar{\psi}^{jb} = 0$$

in the bosonic limit amount the following equation of motion for the scalar field $q = q|_{\theta=0}$

$$\frac{d}{dt}\left[\frac{\dot{q}}{\sqrt{1 - 4\dot{q}\dot{\bar{q}}}}\right] = 0.$$

This equation follows from the bosonic action

$$S_{bos} = \int dt \left(1 - \sqrt{1 - 4\dot{q}\dot{\bar{q}}}\right)$$

which is a proper action for a particle in $D = 3$ space-time.

Outline

Despite the explicit construction of the proper equations of motion within superfields version of the coset approach it is poorly adapted for the construction of the action. That is why we use the component version of the coset approach to construct the action. In the application to the present case, the basic steps of this method can be formulated as follows:

- First, on-shell our $N = 8$ supermultiplet $\{q, \bar{q}\}$ contains the following physical components:

$$q = q|_{\theta=0}, \quad \bar{q} = \bar{q}|_{\theta=0}, \quad \psi_{ia} = \psi_{ia}|_{\theta=0}, \ \bar{\psi}^{ia} = \bar{\psi}^{ia}|_{\theta=0}.$$

  They are just the first components of the superfield parameterizing the coset

$$g = e^{itP}\, e^{\theta_{ia}Q^{ia}+\bar{\theta}^{ia}\overline{Q}_{ia}}\, e^{i(qZ+\bar{q}\overline{Z})}\, e^{\psi_{ia}S^{ia}+\bar{\psi}^{ia}\overline{S}_{ia}}.$$

- Second, with respect to broken supersymmetry $\delta\theta = \delta\bar{\theta} = 0$. This means, that the transformation properties of the physical components $q, \bar{q}, \psi, \bar{\psi}$ under broken supersymmetry can be extracted from the coset

$$g|_{\theta=0} = e^{itP}\, e^{i(qZ+\bar{q}\bar{Z})}\, e^{\psi_{ia}S^{ia}+\bar{\psi}^{ia}\bar{S}_{ia}}.$$

With respect to broken supersymmetry the fields $q, \bar{q}$ are just “matter fields”, because $\delta_S q = \delta_S \bar{q} = 0$, while the fermions $\psi_{ia}, \bar{\psi}^{ia}$ are just Goldstone fermions. The fermions $\psi_{ia}, \bar{\psi}^{ia}$ may enter the action trough ein-bein $\mathcal{E}$ or through the covariant derivatives $\mathcal{D}_t q, \mathcal{D}_t \bar{q}$ only, with

$$\partial_t = \mathcal{E}\mathcal{D}_t, \quad \mathcal{E} = E|_{\theta=0} = 1 - i\left(\psi_{ia}\dot{\bar{\psi}}^{ia} + \bar{\psi}^{ia}\dot{\psi}_{ia}\right),$$

$$\mathcal{E}^{-1} = 1 + i\left(\psi_{ia}\mathcal{D}_t\bar{\psi}^{ia} + \bar{\psi}^{ia}\mathcal{D}_t\psi_{ia}\right).$$

Thus, the unique candidate to be the component on-shell action, invariant with respect to spontaneously broken S supersymmetry reads

$$S = \alpha\int dt + \int dt\,\mathcal{E}\mathcal{F}\left[\mathcal{D}_t q \mathcal{D}_t \bar{q}\right]$$

with arbitrary, for the time being, function $\mathcal{F}$ and constant parameter α.

- Finally, considering the bosonic limit of the supersymmetric action and comparing it with the known bosonic action

$$S_{bos} = \int dt \left(1 - \sqrt{1 - 4\dot{q}\dot{\bar{q}}}\right)$$

one may find the function $\mathcal{F}$:

$$\int dt \left(\alpha + \mathcal{F}\left[\dot{q}\dot{\bar{q}}\right]\right) = \int dt \left(1 - \sqrt{1 - 4\dot{q}\dot{\bar{q}}}\right)$$

$$\Rightarrow \quad \mathcal{F} = \left(1 - \alpha - \sqrt{1 - 4\dot{q}\dot{\bar{q}}}\right) .$$

Therefore, the most general component action possessing the proper bosonic limit and invariant under spontaneously broken supersymmetry has the form

$$S = \alpha \int dt + (1-\alpha) \int \mathcal{E} dt - \int \mathcal{E} dt \sqrt{1 - 4\mathcal{D}_t q \mathcal{D}_t \bar{q}} .$$

The final step is to check the invariance of the action under unbroken Q supersymmetry which is realized on the components as follows

$$\delta_Q^* q = -2\epsilon^{ia}\psi_{ia} + i\left(\epsilon^{ia}\psi_{ia}\bar{\lambda} + \bar{\epsilon}^{ia}\bar{\psi}_{ia}\lambda\right)\partial_t q,$$

$$\delta_Q^* \psi_{ia} = \bar{\epsilon}_{ia}\lambda + i\left(\epsilon^{jb}\psi_{jb}\bar{\lambda} + \bar{\epsilon}^{jb}\bar{\psi}_{jb}\lambda\right)\partial_t \psi_{ia} \,.$$

Here, λ is the first component of the corresponding superfield λ defined as

$$\lambda = \frac{2i\mathcal{D}_t q}{1+\sqrt{1-4\mathcal{D}_t q \mathcal{D}_t \bar{q}}} \,.$$

As a result, we find that the unique component action, invariant under both unbroken Q and broken S $N = 8$ supersymmetries reads

$$S = 2\int dt - \int \mathcal{E} dt \left(1+\sqrt{1-4\mathcal{D}_t q \mathcal{D}_t \bar{q}}\right) \,.$$

Outline

It is almost evident, that the universality of the action can be used to extend our construction to the cases of $N = 4 \cdot 2^k$ supersymmetries by adding the needed numbers of $SU(2)$ indices to the superscharges as

$$Q \to Q^{\alpha_1 \cdots \alpha_k}, \; \overline{Q} \to \overline{Q}_{\alpha_1 \ldots \alpha_k},$$
$$S \to S^{\alpha_1 \cdots \alpha_k}, \; \overline{S} \to \overline{S}_{\alpha_1 \ldots \alpha_k},$$

obeying the $N = 4 \cdot 2^k$ Poincaré superalgebra

$$\left\{ Q^{\alpha_1 \cdots \alpha_k}, \overline{Q}_{\beta_1 \ldots \beta_k} \right\} = 2\delta^{\alpha_1}_{\beta_1} \ldots \delta^{\alpha_k}_{\beta_k} P \,, \quad \left\{ S^{\alpha_1 \cdots \alpha_k}, \overline{S}_{\beta_1 \ldots \beta_k} \right\} = 2\delta^{\alpha_1}_{\beta_1} \ldots \delta^{\alpha_k}_{\beta_k} P \,,$$
$$\left\{ Q^{\alpha_1 \cdots \alpha_k}, S^{\beta_1 \cdots \beta_k} \right\} = 2i\epsilon^{\alpha_1 \beta_1} \ldots \epsilon^{\alpha_k \beta_k} Z \,,$$
$$\left\{ \overline{Q}_{\alpha_1 \ldots \alpha_k}, \overline{S}_{\beta_1 \ldots \beta_k} \right\} = -2i\epsilon_{\alpha_1 \beta_1} \ldots \epsilon_{\alpha_k \beta_k} \overline{Z} \,.$$

Again, the component action describing super particles in $D = 3$ space with $N = 4 \cdot 2^k$ Poincaré supersymmetry partially broken down to $N = 2 \cdot 2^k$ one will be given by the same expression with the following substitutions

$$\psi \to \psi_{\alpha_1 \ldots \alpha_k}, \; \bar{\psi} \to \bar{\psi}^{\alpha_1 \cdots \alpha_k}, \; \mathcal{E} = 1 - i \left(\psi_{\alpha_1 \ldots \alpha_k} \dot{\bar{\psi}}^{\alpha_1 \cdots \alpha_k} + \bar{\psi}^{\alpha_1 \cdots \alpha_k} \dot{\psi}_{\alpha_1 \ldots \alpha_k} \right).$$

We can also apply this approach to construct a model of superparticle in $D = 5$, which possesses 8 manifest and 8 spontaneously broken supersymmetries, starting with the following superalgebra

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon^{ij}\Omega_{\alpha\beta}P, \quad \{S^{a\alpha}, S^{b\beta}\} = -\epsilon^{ab}\Omega^{\alpha\beta}P,$$

$$\{Q^i_\alpha, S^{b\beta}\} = \delta^\beta_\alpha Z^{ib}, \quad \{i, a = 1, 2; \alpha, \beta = 1, 2, 3, 4\},$$

where the invariant $Spin(5)$ symplectic metric $\Omega_{\alpha\beta}$, allowing to raise and lower the spinor indices, obeys the conditions

$$\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}\,, \quad \Omega^{\alpha\beta} = -\frac{1}{2}\,\epsilon^{\alpha\beta\lambda\sigma}\Omega_{\lambda\sigma}\,, \quad \Omega_{\alpha\beta} = -\frac{1}{2}\,\epsilon_{\alpha\beta\lambda\sigma}\Omega^{\lambda\sigma}\,, \quad \Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta^\gamma_\alpha\,.$$

We present here the final result of the component action in this case

$$S = \int dt \left[2 - \mathcal{E}\left(1 + \sqrt{1 - 2\mathcal{D}_t q^{ia}\mathcal{D}_t q_{ia}}\right)\right],$$

$$\partial_t = \mathcal{E}\,\mathcal{D}_t\,, \quad \mathcal{E} = 1 + \frac{1}{2}\,\Omega^{\beta\gamma}\psi^a_\beta\partial_t\psi_{a\gamma}\,.$$

We proposed a method to construct the on-shell component actions for theories with $1/2$ partial breaking of global supersymmetry within the nonlinear realization approach.

In contrast with the standard superfield approach in which unbroken supersymmetry plays the leading role, we have shifted the attention to the spontaneously broken supersymmetry.

It turns out that in the theories in which half of supersymmetries are spontaneously broken, all physical fermions are just the fermions of the nonlinear realization. Moreover, the transformation properties of these fermions with respect to the broken supersymmetry are the same as in the famous Volkov-Akulov model.

Just this fact completely fixed all possible appearances of the fermions in the component action.

It should be clear that the extremely simple form of the component actions is achieved due to the quite special choice of the physical components: all of them are fields of the nonlinear realization.
This is in a dramatic contrast with the superfield approach, in which the main objects are the superfields of the linearly realized broken supersymmetry.
Of course, it is preferable to have the superfield actions, but their very nice superspace forms become very complicated after passing to components.
Moreover, in such component actions it is a very nontrivial task to select some geometric objects and structures. In this respect, our construction looks as an alternative one, and the component form and on-shell character of the actions is the price we have to pay for their simplicity and clear geometric meaning.

It seems that the unique serious limitation of our construction is its validity for theories with $1/2$ breaking of the global supersymmetries only.
But even if it is so, the one explicit example given here is not enough to prove the efficiency of our approach.
The method was already applied to the construction of the component supermembrane action in $D = 5$ with $N = 4 \to N = 2$ breaking of global supersymmetry
S.Bellucci, N.Kozyrev, S.Krivonos and A.Yeranyan,
Supermembrane in D=5: component action, arxiv:1312.0231.
But we think that one of the most interesting cases are those for which the components actions are still unknown (the action for partial breaking of $N = 1, D = 10$ supersymmetry with the hypermultiplet as the Goldstone superfield , etc.).
Among the most complicated and urgent task is the construction of $N = 2$ Born-Infeld action within our scheme.